Efficient, strategy-proof and almost budget-balanced assignment

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Abstract

Call a Vickrey-Clarke-Groves (VCG) mechanism to assign p identical objects among n agents, *feasible* if cash transfers yield no deficit. The *efficiency loss* of such a mechanism is the *worst* (largest) ratio of the budget surplus to the efficient surplus, over all profiles of non negative valuations. The optimal (smallest) efficiency loss $\widehat{L}(n,p)$ satisfies $\widehat{L}(n,p) \leq \widehat{L}(n,\{\frac{n}{2}\}) \leq \frac{4}{3\sqrt{n}}$ for all n. If $\frac{p}{n}$ is strictly smaller or strictly larger than $\frac{1}{2}$, the convergence of $\widehat{L}(n,p)$ to zero is exponential in n.

The optimal mechanism achieving $\widehat{L}(n,p)$ is individually rational (participation is voluntary) if $p=1$, but not if $p \geq 2$. Among all feasible and voluntary mechanisms, the optimal efficiency loss $L^*(n,p)$ is not significantly larger than $\widehat{L}(n,p)$ if $\frac{p}{n} \leq 2$. But it does not converge to zero if $\frac{p}{n} > 2$.

1 The problem and the punch line

A group of n agents must assign p *identical* objects, where $p < n$. These objects are desirable so we talk of "goods" and we have a simple *rationing* problem: each agent claims a unit but all claims can't be met. For instance the n friends share p tickets to a popular show; the city assigns a few alcohol licenses to bar owners; a school principal distributes scarce computers to classrooms, or a manager distributes fancy machines in short supply between several divisions. Cash transfers are used to compensate the losers (who get no object) from the winner's pocket, and to align incentives and efficiency.

Preferences are quasi-linear in money, described by a non negative *valuation* (willingness to pay) for an object. A Vickrey-Clarke-Groves (VCG) mechanism performs monetary transfers inducing truthful revelation of individual valuations and implements an efficient assignment, i.e., assigns the objects to the p

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agents with the highest valuations. Cash transfers don't balance out at some profiles of valuations, therefore to run the mechanism the participants must find a *residual claimant* who will either 'burn' the surplus of money or cover the deficit. In the above examples, the residual claimant is the entity responsible for distributing the objects.

We consider VCG mechanisms that are *self-sufficient* (feasible): money may flow out but not in, in other words the residual claimant is never called to cover a deficit. We measure the surplus loss at a profile of valuations by the ratio of the budget surplus (the money burnt) to the efficient surplus; we measure the overall performance of a mechanism as the *worst* such ratio over all possible profiles of valuations, and call this number the *efficiency loss* of the mechanism.

We compute for all n and p the smallest possible efficiency loss $\widehat{L}(n,p)$ among all feasible VCG mechanisms, and a canonical mechanism achieving $\widehat{L}(n,p)$. We also compute the minimal efficiency loss $L^*(n,p)$ and an optimal mechanism, under the additional constraint of *Voluntary Participation* (individual rationality) requiring that no participant ends up with a net loss. We call such mechanisms *voluntary.* In the fair division interpretation of our model the objects are the common property of the participants, thus unvoluntary mechanisms are unacceptable. Not so if the benevolent manager is mandated to eliminate wasteful dissipation of money and has the power to tax an agent who receives no object and charge one who is served more than her valuation. In this latter context our optimal (perhaps unvoluntary) mechanisms are at least fair under the veil of ignorance, i.e., they treat all agents symmetrically. Our results cover both contexts, and illuminate some important limitations of the voluntary participation constraint.

If we assign a single object, $p = 1$, the optimal voluntary mechanism is also optimal among unvoluntary ones. Moreover it achieves asymptotic efficiency at exponential speed in n: $\widehat{L}(n,1) = L^*(n,1) = \frac{n-1}{2^{n-1}-1}$. For instance with 10 agents the budget surplus *never* exceeds 1.8% of the efficient surplus. We illustrate the construction of this mechanism when $n = 5$ (and $p = 1$). Start from the Vickrey auction, where the residual claimant sells the object at the second highest price. In addition to the Vickrey transfers, all agents, including the winner of the object, receive a cash *rebate* – a share of the auctioneer's revenue – that is independent of their own bid hence preserves the incentive to bid truthfully. The challenge is to choose the rebates small enough that feasibility is preserved, but large enough that they absorb most of the auctioneer's revenue.

Write a_i for agent i's valuation and a^{*k} for the k-th highest valuation. Agent i either gets the object at the second highest price a^{*2} or gets no object and no money. Suppose that, in addition, she gets the rebate $r(a_{-i}) = \frac{1}{5}a^{*2}_{-i}$ where a^{*2}_{-i} is the second highest price *among all valuations other than her own.* Then the mechanism distributes $\frac{1}{5}a^{*3}$ to the two highest valuation agents and $\frac{1}{5}a^{*2}$ to the other three, so that total payment by the agents is

$$a^{*2} - \frac{2}{5}a^{*3} - \frac{3}{5}a^{*2} = \frac{2}{5}(a^{*2} - a^{*3}) \geq 0$$

and feasibility is preserved. The corresponding efficiency loss is 40%[1]. But we can do better: Theorem 1 in section 5 states that the optimal choice of rebates is

$$r(a_{-i}) = \frac{11}{45}a^{*2}_{-i} - \frac{1}{9}a^{*3}_{-i} + \frac{1}{15}a^{*4}_{-i}$$

and shows that the efficiency loss of this mechanism is $L^*(5,1) = \frac{4}{15} = 27\%$.

If we must assign two or more objects, the voluntary participation constraint has bite: $\widehat{L}(n,p) < L^*(n,p)$ for all $p \geq 2$. The most dramatic illustration is for $p = n-1$, where no voluntary mechanism improves upon the Vickrey auction and $L^*(n,n-1) = 1$,so that at some profiles the entire surplus goes to the residual claimant. But if we can use unvoluntary mechanisms we get $\widehat{L}(n,n-1) = \frac{n-1}{n2^{n-2}-1} \simeq \frac{1}{2^{n-2}}$.

One of our main results is that for unvoluntary mechanisms, asymptotic efficiency holds uniformly in p and the worst choice of p is the integer $\frac{n}{2}$ or $\frac{n-1}{2}$, denoted $\{\frac{n}{2}\}$:

$$\max_{1\leq p\leq n-1} \widehat{L}(n,p) = \widehat{L}(n,\{\frac{n}{2}\}) \simeq \frac{0.8}{\sqrt{n}}$$

The critical value of the ratio $\frac{p}{n}$, that we call the *scarcity index* of the assignment problem, is $\frac{1}{2}$. Loosely speaking, if $p < \frac{n}{2}$ the two optimal efficiency losses $\widehat{L}(n,p)$ and $L^*(n,p)$ vanish exponentially fast in n. On the other hand if $p > \frac{n}{2}$ no voluntary mechanism is asymptotically efficient ($L^*(n,p)$ remains bounded away from zero), whereas among unvoluntary mechanisms efficiency can still be achieved exponentially fast.

The mathematical backbone of our results in the one object case is the approximation of the function $\max_i\{x_i\}$ for $x \geq 0$, by an additively decomposable function of the form $\sum_i g_i(x_{-i})$. We require an approximation from above and measure the approximation error by the maximal ratio $\frac{\sum_i g_i(x_{-i})}{\max_i\{x_i\}}$. We find the optimal approximation. For the p objects case, we solve two similar approximation problems (with or without the voluntary participation constraint) for the function $\sum_{k=1}^{p} x^{*k}$ where x^{*k} is the k-th highest among the n variables x_i.

2 Relation to the literature

The general result of [18] applies to our model: the VCG mechanisms are characterized by strategyproofness and allocative efficiency. These mechanisms cannot be budget balanced at all profiles ([17]), so a small set of papers evaluate the asymptotic behavior of their budget imbalance under specific Bayesian assumptions on the distribution of utilities[2]. When asymptotic efficiency prevails, it relies on the heroic assumption that types are independently and identically distributed; in more general contexts computations are typically untractable.

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[1] If we only allow rebates depending upon the highest three valuations this is actually optimal: Proposition 2 in section 6.

[2] In the provision of public goods, references [17], [9] and [42] evaluate the pivotal mechanism and other VCG mechanisms, while [13] examine more general strategyproof mechanisms allowing a degree of allocative inefficiency; see also [26] for the exchange of private goods.

All VCG mechanisms are prior-free (they do not make use of statistical information about types), but their asymptotic efficiency properties in the papers just mentioned are not. In this paper the efficiency of a mechanism is evaluated by its *worst case* performance, thus it is prior-free as well. The worst case analysis is commonplace in the Operations Research and Computer Science literatures where it is often referred to as *competitive analysis* (e.g., [39]). It plays a central role in the algorithmic approach to mechanism design[3]. Although less familiar, the worst case analysis is not without precedent in the micro-economic literature on VCG mechanisms: [27] and [10] use it to discuss the pivotal mechanism in the public good provision problem and [31] does the same in a cost sharing problem.

The idea of refunding part of the budget surplus of the pivotal mechanism while respecting incentives goes back to [25], and was developed by [3] for the public good provision problem. It was recently used in the assignment problem by Cavallo [5] and Guo and Conitzer [14], who also apply the worst case analysis. They use a different performance index, namely the ratio of the budget surplus to the revenue of the Vickrey auctioneer: this revenue is not a good proxy for the potential welfare gains in the assignment economy because it can be arbitrarily smaller than the efficient surplus, therefore the interpretation of their performance index is unclear. Reference [5] proposes a rebate that only depends upon the $p+2$ highest bids and belongs to the family of mechanisms identified in our Proposition 2; [14] discovers independently the linear optimal mechanism of Theorem 1, and offer an alternative characterization limited to the class of linear rebates (while Theorem 1 below applies to all VCG mechanisms). Another difference is that the alternative performance index artificially eliminates the non voluntary mechanisms of our Theorem 2, that remain asymptotically efficient when p is larger than $\frac{n}{2}$ (Theorem 3).

Two more papers applying VCG mechanisms to the assignment of p identical *bads* (tasks) follow a different yet related route. Porter, Shoam and Tennenholtz ([33]) propose an original test of equity called k-fairness (discussed in subsection 5.3; see also [2]) that leads to their 3-Fair mechanism. In the one object case, this is the mechanism introduced in [5] (see Proposition 2). However when $p \geq 2$, the 3-Fair mechanism is unvoluntary and is not comparable to the mechanisms in our Theorem 2.

3 Contents

We set the model in section 4. Our main results in subsection 5.1 give the optimal efficiency performance $L^*(n,p)$ of voluntary and feasible VCG mechanisms and an optimal mechanism with linear rebates (Theorem 1); then the

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[3] It is used to evaluate the competitiveness of Nash equilibrium behavior in congestion problems on a network ([23], [37], [1], [34]), and in one-dimensional cost sharing problems ([19], [20], [21], [38], [41], [6], [29], [30]); to discuss the tradeoff between budget balance and allocative efficiency for (group) strategyproof cost sharing mechanisms ([31], [35], [36], [22]); and to design revenue maximizing strategyproof auction mechanisms ([4], [16]).

optimal efficiency performance $\widehat{L}(n,p)$ of feasible VCG mechanisms and an optimal mechanism with linear rebates (Theorem 2). Theorem 3 in subsection 5.2 describes the asymptotic efficiency, or lack thereof, of these two families of optimal mechanisms. Proposition 1 in subsection 5.3 evaluates the optimal performance $L^{\#}(n,p)$ of VCG mechanisms in which the residual claimant may have to subsidize the mechanism (in which case the budget deficit is interpreted as the cost of running the mechanism; see section 4).

We apply the *fair division* viewpoint to the optimal voluntary mechanisms of Theorem 1 in section 6. They treat equals equally, and even meet the stronger *anonymity* property. However they generate *envy* because they typically give different cash rebates to the inefficient agents who get no object; and the efficient agents may envy the inefficient ones as well.

In section 7 we assume that our mechanism can only keep track of the $p+m$ highest valuations[4]. Proposition 2 generalizes Theorem 1: it gives the corresponding optimal efficiency performance $L^*(n,p,m)$ and an optimal mechanism with linear rebates.

Section 8 briefly discusses two possible generalizations of our results. First to the class of all strategyproof mechanisms that may not always assign the objects efficiently: it is easy to give examples where some of these non VCG mechanisms have a smaller efficiency loss than $L^*(n,p)$, or even than $\widehat{L}(n,p)$. Next to the assignment of p *heterogenous* objects. Section 9 contains the proofs.

4 The model

We have p **identical** objects and a set N of n agents, who each need at most one object. Assume a rationing situation, i.e., $n > p \geq 1$ (if $n \leq p$ everyone gets an object and goes home). Monetary transfers are feasible and agent i's value for an object (relative to no object) is a_i.

Valuations are private information and must be revealed truthfully for incentive compatibility. The only restriction is $a_i \geq 0$: objects are weakly desirable. We use the following notation. Given a profile of valuations $a \in \mathbb{R}_+^N$, the vector $a^* \in \mathbb{R}_+^n$ is its permutation where coordinates are arranged decreasingly

$$a^{*1} \geq a^{*2} \geq \cdots \geq a^{*n} \tag{1}$$

For any $i \in N$, the $(N \diagdown \{i\})$-profile a_{-i} obtains by deleting the i-th coordinate, and a^*_{-i} denotes its permutation by weakly decreasing coordinates. The efficient surplus given p objects and the profile of valuations a is

$$v_p(a) = a^{*1} + \cdots + a^{*p} \tag{2}$$

Similarly $v_p(a_{-i}) = a^{*1}_{-i} + \cdots + a^{*p}_{-i}$ is the efficient surplus in the absence of agent i.

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[4] Thanks to Olivier Gossner for suggesting this variant.

A general VCG mechanism ([17]) is defined by n arbitrary real valued functions h_i on $\mathbb{R}_+^{N\smallsetminus\{i\}}$. The function h_i determines agent i's net utility U_i

$$U_i(a) = v_p(a) - h_i(a_{-i}) \text{ for all } a \in \mathbb{R}_+^N \tag{3}$$

At a profile a, the mechanism assigns an object to an efficient subset of p agents. Monetary compensations are adjusted so that the net utility of every agent, whether or not she gets an object, is given by (3). Recall that the tie breaking rule (if there is more than one efficient subset, i.e., $a^{*p} = a^{*(p+1)}$) is arbitrary, it affects neither welfare nor incentives.

We denote by Δ the budget imbalance of mechanism (3):

$$\Delta(a) = v_p(a) - \sum_{i\in N} U_i(a) = \sum_{i\in N} h_i(a_{-i}) - (n-1)v_p(a)$$

In the most general VCG mechanisms the sign of Δ is arbitrary: money can flow in or out and the residual claimant will absorb any surplus or cover any deficit. It is sometimes realistic to allow a budget imbalance $\Delta(a)$ of arbitrary sign. As long as the subsidy needed to balance the transfers remains small with respect to the total surplus captured by the mechanism, its strong multiplicative effect vindicates the emergence of a benevolent residual sponsor.

The performance of a mechanism is measured by the following index:

$$L(n,p) = \max \frac{\text{money imbalance}}{\text{efficient surplus}} = \max_{a\in\mathbb{R}_+^N\smallsetminus\{0\}} \frac{|\Delta(a)|}{v_p(a)} \tag{4}$$

If $\Delta(a) > 0$ the residual claimant receives part of the efficient surplus in cash, so we can interpret $\frac{|\Delta(a)|}{v_p(a)}$ as a relative efficiency loss. This interpretation is not valid if $\Delta(a) < 0$, because in this case the residual claimant subsidizes the participants; we then regard $\frac{|\Delta(a)|}{v_p(a)}$ as the relative cost of running the mechanism, and $\frac{v_p(a)}{|\Delta(a)|}$ as the multiplicator effect of the subsidy.

For most of the paper we restrict attention to *self-sufficient* mechanisms where the money only flows out. We call these mechanisms *feasible*:

$$\textit{Feasibility } (\text{F}){:}\ \Delta(a) \geq 0 \Leftrightarrow \sum_{i\in N} h_i(a_{-i}) \geq (n-1)v_p(a) \text{ for all } a \tag{5}$$

Under Feasibility, we call the index $L(n,p)$ the *worst efficiency loss,* or simply the *efficiency loss* of the mechanism:

$$L(n,p) = \max \frac{\text{outflow of money}}{\text{efficient surplus}} = \max_{a\in\mathbb{R}_+^N\smallsetminus\{0\}} \frac{\Delta(a)}{v_p(a)} \tag{6}$$

If $\Delta(0) > 0$ it is natural to set $L(n,p) = \infty$, and to modify the definition (6) accordingly. This detail will not create additional constraint on any of the mechanisms we discuss, so we omit to mention it in the sequel.

We call a mechanism *voluntary* if no one ever suffers a net loss as a result of participating. This requirement, often called *individual rationality,* is compelling when our model is interpreted as a fair division problem**:**

$$\begin{aligned} \textit{Voluntary Participation (VP)} \quad &: \quad U_i(a) \geq 0 \text{ for all } a, \text{ all } i \\ &\Leftrightarrow \quad h_i(a_{-i}) \leq v_p(a_{-i}) \text{ for all } a_{-i}, \text{ all } i \end{aligned} \tag{7}$$

Note that F and VP imply $0 \leq L(n,p) \leq 1$. Indeed inequality (7) gives $h_i(a_{-i}) \leq v_p(a_{-i}) \leq v_p(a)$ for all i, which sums to $\Delta(a) \leq v_p(a)$.

A benchmark mechanism is the *Vickrey auction* (also called *pivotal mechanism* see [17]), in which the residual claimant "owns" the objects and sells them at the $(p+1)$st highest price. Thus $h_i^{vick}(a_{-i}) = v_p(a_{-i})$ and

$$U_i^{vick}(a) = v_p(a) - v_p(a_{-i}) \text{ for all } i \text{ and } a \tag{8}$$

The Vickrey auction is feasible and (barely) induces voluntary participation: an inefficient agent breaks even, an efficient one gets $U_i^{vick}(a) = a_i - a^{*(p+1)}$. The residual claimant captures the whole surplus if $a_1^* = \cdots = a_{p+1}^*$, implying $L^{vick}(n,p) = 1$. The Vickrey auction has the largest efficiency loss among all feasible and voluntary VCG mechanisms.

In view of definition (8), inequality (7)) reads $U_i^{vick}(a) \leq U_i(a)$ for all i and a: a VCG mechanism is voluntary *if and only if* it is Pareto superior to the Vickrey auction.

Whether or not the mechanism under consideration is voluntary, it will be convenient to write the functions $h_i(a_{-i})$ in (3) as $h_i(a_{-i}) = v_p(a_{-i}) - r_p(i; a_{-i})$ where $r_p(i; a_{-i})$ is a *rebate function.* Hence the general form of VCG mechanisms in our model:

$$U_i(a) = v_p(a) - v_p(a_{-i}) + r_p(i; a_{-i}) = U_i^{vick}(a) + r_p(i; a_{-i}) \text{ for all } a \in \mathbb{R}_+^N \tag{9}$$

When Voluntary Participation holds, and only then, we interpret $r_p(i; a_{-i})$ as agent i's share of the seller's revenue in the Vickrey auction.

5 Main results

5.1 Optimal feasible mechanisms: voluntary and unvoluntary

Let $\binom{s}{k}$ be the binomial coefficient "take k among s". For any integers such that $t \leq t' \leq s$ we define

$$B_s^{t,t'} = \sum_{k=t}^{t'} \binom{s}{k}, \quad B_s^{t\rightarrow} = B_s^{t,s}, \quad B_s^{\rightarrow t} = B_s^{0,t} \tag{10}$$

Theorem 1 (*under F and VP)*

Among all feasible and voluntary VCG mechanisms (9), the smallest efficiency loss (6) is

$$L^*(n,p) = \frac{\binom{n-1}{p}}{B_{n-1}^{p\to}} \tag{11}$$

The following linear rebate functions define an optimal mechanism

$$r_p^*(a_{-i}) = \sum_{k=p+1}^{n-1} (-1)^{k-p-1} \frac{pL^*(n,p)}{kL^*(n,k)} a_{-i}^{*k} \text{ if } p \leq n-2;\ r_{n-1}^*(a_{-i}) = 0 \tag{12}$$

The corresponding budget surplus is

$$\Delta^*(a) = pL^*(n,p)\{\sum_{k=p+1}^{n} (-1)^{k-p-1} a^{*k}\} \tag{13}$$

Theorem 2 (*under F only*)
Among all feasible VCG mechanisms (9) the smallest efficiency loss $\widehat{L}(n,p)$ (6) is

$$\widehat{L}(n,1) = L^*(n,1)\ ;\ \widehat{L}(n,p) = \frac{\binom{n-1}{p}}{B_{n-1}^{p\to} + \frac{n}{p} B_{n-2}^{\to(p-2)}} \text{ if } 2 \leq p \leq n-1 \tag{14}$$

The following linear rebate functions define an optimal mechanism

$$\widehat{r}_1(a_{-i}) = r_1^*(a_{-i})$$

$$\widehat{r}_p(a_{-i}) = \widehat{L}(n,p)\{\sum_{k=1}^{p-1} \gamma_k a_{-i}^{*k}\} + (1 - \frac{\widehat{L}(n,p)}{L^*(n,p)}) a_{-i}^{*p} + \sum_{k=p+1}^{n-1} (-1)^{k-p-1} \frac{p\widehat{L}(n,p)}{kL^*(n,k)} a_{-i}^{*k} \tag{15}$$

$$\gamma_k = -\frac{n}{n-1} \frac{B_{n-2}^{(n-k)\to}}{\binom{n-2}{n-k-1}} - \frac{1}{n-1} \text{ if } p-k \text{ is odd};\ \gamma_k = \frac{n}{n-1} \frac{B_{n-2}^{(n-k)\to}}{\binom{n-2}{n-k-1}} \text{ if } p-k \text{ is even}$$

(the right summation in (15) is zero if $p = n-1$*).The budget surplus is*

$$\widehat{\Delta}(a) = \widehat{L}(n,p)\{\sum_{k=1,3,\cdots}^{\leq p-1} (p-k)(a^{*(p-k)} - a^{*(p-k+1)}) + p \sum_{k=p+1}^{n} (-1)^{k-p-1} a^{*k}\} \tag{16}$$

The rebate functions (12) and (15) are not the only choices of $r_p(a_{-i})$ achieving, respectively, $L^*(n,p)$ and $\widehat{L}(n,p)$. They are the only choices if we restrict attention to symmetric rebate functions, linear in the a_{-i}^{*k}: see the argument at the end of the proof of Theorems 1,2.

We discuss first the cases of $p = 1$ and $p = (n-1)$. For $p = 1$ Voluntary Participation comes free: the optimal linear rebates under F define a voluntary mechanism, therefore the optimal efficiency loss under F is also the optimal loss under F and VP:

$$\widehat{L}(n,1) = L^*(n,1) = \frac{n-1}{2^{n-1}-1} \simeq \frac{2n}{2^n}$$

Contrast this with the situation for $p = n-1$. Then we cannot improve upon the Vickrey auction by a voluntary and feasible VCG mechanism ($L^*(n,n-1) = 1$ and $r^*_{n-1}(a_{-i}) = 0$), whereas the optimal feasible (unvoluntary) mechanism achieves an efficiency loss even smaller than in the one object case:

$$\widehat{L}(n,n-1) = \frac{n-1}{n2^{n-2}-1} \simeq \frac{4}{2^n}$$

We compute first the optimal linear mechanism for $p = 1$ and small values of n. For $n = 2$ the Vickrey auction cannot be improved. For $n = 3,4,5,6$ equation (12) reads:

$$L^*(3,1) = \frac{2}{3} \text{ and } r^*_1(a_{-i}) = \frac{1}{3}a^{*2}_{-i}$$

$$L^*(4,1) = \frac{3}{7} \text{ and } r^*_1(a_{-i}) = \frac{2}{7}a^{*2}_{-i} - \frac{1}{7}a^{*3}_{-i}$$

$$L^*(5,1) = \frac{4}{15} \text{ and } r^*_1(a_{-i}) = \frac{11}{45}a^{*2}_{-i} - \frac{1}{9}a^{*3}_{-i} + \frac{1}{15}a^{*4}_{-i}$$

$$L^*(6,1) = \frac{5}{31} \text{ and } r^*_1(a_{-i}) = \frac{13}{62}a^{*2}_{-i} - \frac{8}{93}a^{*3}_{-i} + \frac{3}{62}a^{*4}_{-i} - \frac{1}{31}a^{*5}_{-i}$$

$$L^*(7,1) = \frac{2}{21} \text{ and } r^*_1(a_{-i}) = \frac{19}{105}a^{*2}_{-i} - \frac{1}{15}a^{*3}_{-i} + \frac{11}{315}a^{*4}_{-i} - \frac{1}{45}a^{*5}_{-i} + \frac{1}{63}a^{*6}_{-i}$$

Equation (12) and the fact that $L^*(n,k)$ increases in k (Theorem 3 below) imply $r^*_1(a_{-i}) = \sum_{k=2}^{n-1} \beta_k a^{*k}_{-i}$ where, the β_k start with $\beta_1 > 0$, alternate in sign and $|\beta_k|$ decreases in k. For an arbitrary $p, 1 \leq p \leq n-2$, the general form of the optimal rebates is similarly $r^*_p(a_{-i}) = \sum_{k=p+1}^{n-1} \beta_k a^{*k}_{-i}$, where the coefficients start positive, alternate in sign and decreases in absolute value.

Turning to the case $p = n-1$, equation (15) (where the second summation disappears) gives:

$$\widehat{L}(3,2) = \frac{2}{5} \text{ and } \widehat{r}_2(a_{-i}) = -\frac{1}{5}a^{*1}_{-i} + \frac{3}{5}a^{*2}_{-i}$$

$$\widehat{L}(4,3) = \frac{1}{5} \text{ and } \widehat{r}_3(a_{-i}) = -\frac{1}{5}a^{*2}_{-i} + \frac{4}{5}a^{*3}_{-i}$$

$$\widehat{L}(5,4) = \frac{4}{39} \text{ and } \widehat{r}_4(a_{-i}) = -\frac{1}{39}a^{*1}_{-i} + \frac{5}{117}a^{*2}_{-i} - \frac{23}{117}a^{*3}_{-i} + \frac{35}{39}a^{*4}_{-i}$$

$$\widehat{L}(6,5) = \frac{1}{19} \text{ and } \widehat{r}_5(a_{-i}) = -\frac{1}{38}a^{*2}_{-i} + \frac{1}{19}a^{*3}_{-i} - \frac{7}{38}a^{*4}_{-i} + \frac{18}{19}a^{*5}_{-i}$$

$$\widehat{L}(7,6) = \frac{6}{223} \text{ and}$$

$$\widehat{r}_6(a_{-i}) = -\frac{1}{223}a_{-i}^{*1} + \frac{7}{1115}a_{-i}^{*2} - \frac{26}{1115}a_{-i}^{*3} + \frac{56}{1115}a_{-i}^{*4} - \frac{187}{1115}a_{-i}^{*5} + \frac{217}{223}a_{-i}^{*6}$$

The pattern is now $\widehat{r}_{n-1}(a_{-i}) = \sum_{k=1}^{n-1} \gamma_k a_{-i}^{*k}$ where the γ_k alternate in sign, $|\gamma_k|$ increase in absolute value, and γ_{n-1} is positive and slightly below 1; if n is even $\gamma_1 = 0$ and $\gamma_2 < 0$ and if n is odd $\gamma_1 = -\frac{\widehat{L}(n,n-1)}{n-1}$.

Comparing (14) and (11) we see that if $2 \leq p \leq n-1$, allowing unvoluntary mechanisms strictly decreases the optimal efficiency loss: $\widehat{L}(n,p) < L^*(n,p)$. In the rebates $\widehat{r}_p(a_{-i})$ ((15)) the first p terms make a sum $\sum_{k=1}^{p} \gamma'_k a_{-i}^{*k}$ similar to that for $\widehat{r}_{n-1}(a_{-i})$ (coefficients increasing in absolute value, alternating in sign and $\gamma_p > 0$) while the next $(n-1-p)$ terms are just $\frac{\widehat{L}(n,p)}{L^*(n,p)} r_p^*(a_{-i})$. For instance

$$\widehat{L}(6,3) = \frac{5}{13} \text{ and } \widehat{r}_3(a_{-i}) = -\frac{3}{26}a_{-i}^{*2} + \frac{5}{13}a_{-i}^{*3} + \frac{9}{26}a_{-i}^{*4} - \frac{3}{13}a_{-i}^{*5}$$

where the first two terms are $\sum_{k=1}^{p} \gamma'_k a^{*k}$ and the last two are $\frac{8}{13} r_3^*(a_{-i})$.

5.2 Asymptotic efficiency

The scarcity ratio $\frac{p}{n}$ essentially determines the asymptotic behavior of $\widehat{L}(n,p)$ and $L^*(n,p)$ for large n.

We use the notation $f(n) \simeq g(n)$ for $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$; the expression "$f(n)$ is exponential" means the existence of $K > 0$ and $\alpha < 1$ such that $f(n) \leq K\alpha^n$ for all n; finally $\{\frac{n}{2}\}$ is the integer $\frac{n}{2}$ or $\frac{n-1}{2}$.

Theorem 3

i) $L^*(n,p)$ *increases strictly in* p*, decreases strictly in* n*;* $\widehat{L}(n,p)$ *increases in* n *for* $p \leq n \leq 2p-1$*, decreases in* n *if* $2p \leq n$*;* $\widehat{L}(n,p)$ *increases in* p *for* $1 \leq p \leq \{\frac{n}{2}\}$*, decreases in* p *if* $\{\frac{n}{2}\} \leq p \leq n$*.*

ii) $\widehat{L}(n,p)$ *converges to zero uniformly in* p*:*

$$\max_{1\leq p\leq n} \widehat{L}(n,p) = \widehat{L}(n,\{\frac{n}{2}\}) \leq \frac{4}{3\sqrt{n}} \text{ for all } n$$

iii) For p *fixed,* $L^*(n,p)$ *and* $\widehat{L}(n,p)$ *are exponential in* n*:* $L^*(n,p) \simeq \widehat{L}(n,p) \simeq \frac{2}{p!}\frac{n^p}{2^n}$.

iv) For any sequence $p_n, n = 1,2,\cdots$ *such that for some* K*,* $\frac{n}{2} - K \leq p_n \leq \frac{n}{2} + K$ *for all* n*:*

$$L^*(n,p_n) \simeq 2\sqrt{\frac{2}{\pi n}} = \frac{1.59\cdots}{\sqrt{n}};\ \widehat{L}(n,p_n) \simeq \sqrt{\frac{2}{\pi n}} = \frac{0.80\cdots}{\sqrt{n}} \tag{17}$$

v) For any sequence $p_n, n = 1,2,\cdots$ *and any positive number* δ

$$\frac{p_n}{n} \leq \delta < \frac{1}{2} \Rightarrow L^*(n,p_n) \text{ and } \widehat{L}(n,p_n) \text{ are exponential in } n \tag{18}$$

$$\frac{p_n}{n} \geq \delta > \frac{1}{2} \Rightarrow \widehat{L}(n, p_n) \text{ is exponential in } n;\ L^*(n, p_n) \geq \frac{2\delta - 1}{\delta} \text{ for all } n \quad (19)$$

Loosely speaking, $\widehat{L}(n,p)$ and $L^*(n,p)$ converge exponentially fast to zero in n if $\frac{p}{n} < \frac{1}{2}$, and as $\frac{1}{\sqrt{n}}$ if $\frac{p}{n} \simeq \frac{1}{2}$. Yet their behavior is very different if $\frac{p}{n} > \frac{1}{2}$: unvoluntary mechanisms still allow exponentially fast efficiency, while voluntary ones preclude asymptotic efficiency altogether.

5.3 Unfeasible mechanisms

Recall that if a VCG mechanism allows money to flow in (runs a deficit), $L(n,p)$ given by (4) is not interpreted as a (relative) efficiency loss, but as the (relative) cost of running the mechanism.

It turns out that imposing feasibility only increases by a factor of 2 the optimal cost/efficiency loss index $L(n,p)$. We write $L^{\#}(n,p)$ for the minimum of $L(n,p)$ among all VCG mechanisms, not necessarily feasible.

Proposition 1
Among all VCG mechanisms (3), the smallest cost index $L^{\#}(n,p)$ *(4) is such that*

$$\frac{1}{2 + \frac{1}{n-1}} \widehat{L}(n,p) \leq L^{\#}(n,p) \leq \frac{1}{2} \widehat{L}(n,p) \text{ for all } n \text{ and } p$$

We do not compute an actual optimal mechanism, which could be difficult. Another interesting open question is to evaluate the optimal cost index (4) among all voluntar VCG mechanisms, not necessarily feasible. A reasonable conjecture is that it is not significantly smaller than $\frac{L^*(n,p)}{2}$.

6 Fair Division interpretation of the voluntary mechanism

In the fair division interpretation of the model the agents have equal property rights over the objects, and mechanisms must be voluntary. We submit now the optimal mechanisms of Theorem 1 to a handful of familiar equity tests.

They meet the basic equity requirement of *anonymity:* the mapping $a \to U^*(a)$ from $\mathbb{R}_+^N$ into itself is symmetric in all variables (as well as continuous). Another fair feature is the fact that all efficient agents are charged the same price $\pi^* = a^{*(p+1)} - r_p^*(a_{-i})$ for an object. Indeed the terms $a_{-i}^{*(p+1)}, \cdots, a_{-i}^{*(n-1)}$ in the sum (12) do not depend on the particular efficient agent i. Moreover π^* is non negative: no one is subsidized to consume one of the scarce objects. To check $\pi^* \geq 0 \Leftrightarrow r_p^*(a_{-i}) \leq a^{*(p+1)}$ if agent i is efficient, recall that $r_p^*(a_{-i})$ is bounded above by the first term in the sum (12):

$$r_p^*(a_{-i}) \leq \frac{pL^*(n,p)}{(p+1)L^*(n,p+1)} a_{-i}^{*(p+1)} = \frac{p}{n-p-1} \frac{B_{n-1}^{(p+1)\to}}{B_{n-1}^{p\to}} a^{*(p+1)} \quad (20)$$

so it remains to show

$$pB_{n-1}^{(p+1)\rightarrow} \leq (n-p-1)B_{n-1}^{p\rightarrow} \Leftrightarrow (2p-n+1)B_{n-1}^{(p+1)\rightarrow} \leq (n-p-1)\binom{n-1}{p}$$

This is clear if $2p+1 \leq n$ and if $2p+1 \geq n$ it follows after routine computations from the upper bound (35) on $B_s^{t\rightarrow}$ (subsection 9.3.1).

We turn to less appealing normative features of our mechanisms. Notice from equation (12) that for two inefficient agents (who get no object) with different valuations, the lists $a_{-i}^{*(p+1)}, \cdots, a_{-i}^{*(n-1)}$ differ, hence they receive typically different cash rebates and one agent envies the other. As the coefficients of the sum (12) alternate in sign, the higher valuation agent may get a lower or a higher cash transfer. Thus the No Envy test fails and utilities are not comonotonic to valuations.

The following knife-edge profile illustrates another unappealing feature:

$$a_1 = \cdots = a_{p+1} = 1, a_{p+2} = \cdots = a_n = 0 \tag{21}$$

In the Vickrey auction, all net utilities are zero. In the optimal mechanism (12), the $p+1$ efficient agents get zero utility as well: $a_{-i}^{*k} = 0$ for all $k \geq p+1$ gives $U_i^* = U_i^{vick} = 0$ (a subset of p such agents get an object at full price, the last one gets no object and no cash). On the other hand one checks easily that each agent $i, i \geq p+2$, receives the share $\frac{1-L(n,p)}{n-p-1}$ of the efficient surplus. Thus the $n-p-1$ "dummy" agents capture most of the surplus created by the $p+1$ efficient agents.

Remark *The Vickrey auction is clearly envy-free. From Papai's characterization of all VCG mechanisms generating no envy ([32]), it is easy to deduce that the Vickrey auction is the only feasible and non envious VCG mechanism of which the efficiency loss is bounded*[5].

Two more normative differences between our optimal mechanism and the Vickrey auction deserve mentioning. In the latter the utility $U_i^{vick}(a)$ of any agent is weakly increasing in p and weakly decreasing in the set of agents N sharing these p objects[6]. The mechanism of Theorem 1 violates both monotonicity properties. We omit the straightforward verification.

One final equity test often discussed in the fair division literature is *Fair Share Guarantee* (FSG) requiring $U_i \geq \frac{p}{n}a_i$ for all i and a, i.e., each participant is guaranteed her expected utility in the event the objects are allocated randomly, irrespective of the valuation profile[7]. It is a stronger requirement

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[5] Papai shows that for a VCG mechanism meeting No Envy the functions h_i take the form $h_i(a_{-i}) = f(a_{-i}^{*1})$, for some function f independent of i, and with derivative between 0 and 1. Feasibility at the null profile implies $f(0) \geq 0$. Next consider the profile $(\varepsilon, 0, \cdots, 0)$. If $f(0) = \gamma > 0$ the budget surplus is $\gamma + (n-1)f(\varepsilon) - (n-1)\varepsilon \geq n\gamma - (n-1)\varepsilon$ hence the relative loss is unbounded for small ε. If $f(0) = 0$, feasibility gives $(n-1)f(\varepsilon) \geq (n-1)\varepsilon$, therefore f is the identity as claimed.

[6] The former is known in the fair division literature as Resource Monotonicity, the latter as Population Monotonicity.(see e.g., [40]). There are efficient solutions of our assignment problem meeting both properties, but all of them fail the No Envy test (see [28]).

[7] Cramton Gibbons and Klemperer ([7]) discuss this property in the one object case; for general assignment problems see [28].

than VP. It turns out that the FSG property is out of reach under feasibility. Assume to the contrary that the feasible VCG mechanism (3) satisfies FSG. Fix an agent i and a $N\diagdown\{i\}$-profile a_{-i}. At the N-profile a such that $a_i = a^{*p}_{-i}$ FSG implies $h_i(a_{-i}) \leq v_p(a_{-i}) - \frac{p}{n}a^{*p}_{-i}$. Apply this inequality for each i at the profile $a_1 = \cdots = a_p = 1, a_{p+1} = \cdots = a_n = 0$:

$$h_i(a_{-i}) \leq p - 1 \text{ for } i = 1, \cdots, p; h_i(a_{-i}) \leq p - \frac{p}{n} \text{ for } i = p+1, \cdots, n$$

On the other hand feasibility requires $\sum_1^n h_i(a_{-i}) \geq p(n-1)$ and we have a contradiction.

The k- *fairness* property of [33] and [2] is an interesting weakening of the Fair Share Guarantee. It requires $U_i(a) \geq \frac{p}{n}a^{*k}$ for all i and a, i.e., each participant is guaranteed an equal share of a proxy surplus computed from the k-th highest valuation. It is not directly comparable to FSG.

7 Variant: recording fewer bids

In the optimal mechanisms of Theorems 1,2 agent i's rebate is a linear combination of the successive valuations $a^{*k}_{-i}, k = p+1, \cdots, n-1$ of the other participants. If the number of agents is very large it may be too costly to keep track of all reported valuations. Under the assumption that a mechanism can only record the $p+m$ highest reported valuations, we compute an optimal improvement of the Vickrey auction and the corresponding smallest efficiency loss $L^*(n,p,m)$. For brevity we do not compute the optimal feasible unvoluntary mechanisms recording only the $p+m$ highest valuations: such a computation is clearly within the reach of our proof techniques.

Now the rebate function $r_p(a_{-i})$ only depends upon the a^{*k}_{-i} up to $k = p+m-1$. Simply truncating the summation in (12) to the first $p+m-1$ terms is not an option because the corresponding mechanism is not feasible[8]. However the optimal choice of truncated linear rebates is a simple generalization of formulas (12).

Proposition 2
Fix $n, p, p \leq n-2$, and m, $1 \leq m \leq n-p$. Among all feasible and voluntary VCG mechanisms (9) recording only the $p+m$ highest valuations*, the smallest efficiency loss (6) is*

$$L^*(n,p,m) = \frac{\binom{n-1}{p}}{B^{p,p+m-1}_{n-1}} \tag{22}$$

The following linear rebate functions $\widetilde{r}^m_p$ define an optimal mechanism:

$$\widetilde{r}^1_p(a_{-i}) = 0; \widetilde{r}^m_p(a_{-i}) = \sum_{k=p+1}^{p+m-1} (-1)^{k-p-1} \frac{pL^*(n,p,m)}{kL^*(n,k,m+p-k)} a^{*k}_{-i} \text{ for } m \geq 2 \tag{23}$$

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[8] E.g., for $p = 1, m = 2, n = 4$ the truncation gives $r^*_1(a_{-i}) = \frac{2}{7}a^{*2}_{-i}$, so at profile $a = (1,1,1,1)$ we get $U_i = \frac{2}{7}$ while $v_1(a) = 1$.

The corresponding budget surplus is

$$\Delta^*(a) = pL^*(n,p,m)\{\sum_{k=p+1}^{p+m}(-1)^{k-p-1}a^{*k}\} \tag{24}$$

For $m = n - p$ this is Theorem 1. For $m = 1$ we can't improve the Vickrey auction and $L^*(n,p,1) = 1$. For $m = 2$ the rebates (23) and optimal performance are:

$$r_p^2(a_{-i}) = \frac{p}{n}a_{-i}^{*(p+1)} \text{ and } L^*(n,p,2) = \frac{p+1}{n}$$

If $p = 1$ this is the mechanism discussed by [5] and the 3-Fair mechanism of [33]. For $m = 3$ a routine computation gives:

$$r_p^3(a_{-i}) = \frac{p}{n^2-(p+1)n+(p+1)(p+2)}\{na_{-i}^{*(p+1)} - (p+1)a_{-i}^{*(p+2)}\}$$

$$L^*(n,p,3) = \frac{(p+1)(p+2)}{n^2-(p+1)n+(p+1)(p+2)}$$

As for the asymptotic behavior of $L^*(n,p,m)$, the most natural idea is to fix p and the recording capacity m while n grows. Then $L^*(n,p,m)$ is strictly decreasing in n (a proof is given in subsection 9.5.1); moreover (22) implies easily

$$L^*(n,p,m) \simeq \frac{(p+m-1)!}{p!}\frac{1}{n^{m-1}}$$

Asymptotic efficiency is still true as in statement *iii*) of Theorem 3, but at the much slower polynomial speed.

8 Two open problems

8.1 Non VCG mechanisms

The family of strategyproof assignment mechanisms contains many non-VCG members that do not always assign the objects efficiently. If $U_i(a)$ denotes as before agent i's net utility, the worst efficiency loss of a feasible mechanism is

$$L(n,p) = \max_{a\in\mathbb{R}_+^N\setminus\{0\}} \frac{v_p(a) - \sum_N U_i(a)}{v_p(a)}$$

The challenge is to compute the optimal value of $L(n,p)$ over all feasible (or all feasible and voluntary) strategyproof mechanisms.

When the scarcity ratio $\frac{p}{n}$ is large enough, it is easy to construct non VCG mechanisms improving upon the efficiency losses $\widehat{L}(n,p)$ and $L^*(n,p)$. Consider the (non anonymous) mechanism picking an arbitrary agent, say agent 1, giving her an object and making her the residual claimant of a voluntary VCG mechanism (e.g., the Vickrey auction) assigning the remaining $p-1$ objects among agents other than 1. This ensures budget-balance and a worst efficiency loss of $\frac{1}{p}$ (the worst case is that the residual claimant has $a_1 = 0$ while the p efficient agents have the same positive valuation)[9]. For $p = \{\frac{n}{2}\}$ and n large enough this improves upon $\widehat{L}(n, \{\frac{n}{2}\}) \simeq \frac{0.6}{\sqrt{n}}$.

8.2 Heterogenous objects

In the general assignement problem we have a set N of n agents and a set P of p desirable objects. Agent i's valuation for object k is an arbitrary non negative number $a_i(k)$. The efficient surplus $v_p(a)$ maximizes $\sum_N a_i(k(i))$ over all feasible assignments. The definition (6) of the worst relative efficiency loss for a feasible VCG mechanism is now

$$\mathcal{L}(n,p) = \max_{a \in \mathbb{R}_+^{N \times P} \setminus \{0\}} \frac{\Delta(a)}{v_p(a)}$$

The agents may view all objects as identical, therefore the optimal value $\mathcal{L}^*(n,p)$ over all feasible and voluntary VCG mechanisms is at least $L^*(n,p)$. Clearly $\mathcal{L}^*(n,1) = L^*(n,1)$, moreover VP and F imply $\mathcal{L}(n,p) \leq 1$ as before. Therefore Theorem 1 determines $\mathcal{L}^*(n,p)$ in the following cases:

$$\mathcal{L}^*(n,1) = \frac{n-1}{2^{n-1}-1}; \mathcal{L}^*(n,p) = 1 \text{ if } p \geq n-1$$

(for $p \geq n$ we use the fact that the matrix of valuations a may be such that only $n-1$ objects are desired).

Computing $\mathcal{L}^*(n,p)$ for $2 \leq p \leq n-2$ appears to be difficult. The computation of the optimal efficiency loss $\widehat{\mathcal{L}}(n,p)$ over all feasible VCG mechanisms is equally open. A general characterization result by Guo and Conitzer [15] may be helpful to reach an answer.

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9 Proofs

9.1 Theorem 1

Step 1.
We check that (12) defines a feasible and voluntary mechanism with budget loss $\widehat{\Delta}$ in (13) and efficiency loss $L^*(n,p)$ in (11). Fix n and p throughout. From (9) the budget imbalance of our mechanism is

$$\Delta^*(a) = \Delta^{vick}(a) - \sum_{i\in N} r_p^*(a_{-i}) = pa^{*(p+1)} - \sum_{i\in N} r_p^*(a_{-i}) \tag{25}$$

As no confusion may arise, in this step we write B_n^p instead of $B_n^{p\rightarrow}$. A straightforward computation from (12) gives

$$\begin{aligned}\sum_{i\in N} r_p^*(a_{-i}) &= \frac{\binom{n-2}{p-1}}{B_{n-1}^p}\{(n-p-1)\frac{B_{n-1}^{p+1}}{\binom{n-2}{p}}a^{*(p+1)} \\ &\quad + \sum_{k=p+2}^{n} (-1)^{k-p}((k-1)\frac{B_{n-1}^{k-1}}{\binom{n-2}{k-2}} - (n-k)\frac{B_{n-1}^{k}}{\binom{n-2}{k-1}})a^{*k}\}\end{aligned}$$

with the convention that for $k=n$ the term $\frac{B_{n-1}^k}{\binom{n-2}{k-1}}$ is zero. For $2 \leq k \leq n-1$ the two combinatorial identities

$$(k-1)\frac{B_{n-1}^{k-1}}{\binom{n-2}{k-2}} - (n-k)\frac{B_{n-1}^{k}}{\binom{n-2}{k-1}} = (n-1)$$

$$p - \frac{\binom{n-2}{p-1}}{B_{n-1}^p}(n-p-1)\frac{B_{n-1}^{p+1}}{\binom{n-2}{p}} = p - p\frac{B_{n-1}^{p+1}}{B_{n-1}^p} = \frac{p\binom{n-1}{p}}{B_{n-1}^p}$$

simplify the expression of $\Delta^*(a)$ to

$$\Delta^*(a) = \frac{p\binom{n-1}{p}}{B_{n-1}^p} a^{*(p+1)} - (n-1)\frac{\binom{n-2}{p-1}}{B_{n-1}^p} \sum_{k=p+2}^{n} (-1)^{k-p} a^{*k}$$

and the proof of (13) is complete because $(n-1)\binom{n-2}{p-1} = p\binom{n-1}{p}$. Feasibility means that $\Delta^*(a)$ is non negative; indeed it is the sum of terms with alternating signs and weakly decreasing absolute value. Clearly $\Delta^*(a) \leq pL^*(n,p)a^{*(p+1)}$ for all a, with equality at the profile (21). Thus the efficiency loss of our mechanism is $L^*(n,p)$.

It remains to check VP, i.e., $r_p^*(a_{-i}) \geq 0$. Property (36) in subsection 9.3.2 states that $\frac{B_s^t}{\binom{s}{t}}$decreases in t. As $\frac{B_s^t}{\binom{s-1}{t-1}} = \frac{s}{t}\frac{B_s^t}{\binom{s}{t}}$ we see that $\frac{B_s^t}{\binom{s-1}{t-1}}$ is also strictly decreasing in t. Thus $\frac{B_{n-1}^k}{\binom{n-2}{k-1}} a_{-i}^{*k}$ is weakly decreasing in k and $r_p^*(a_{-i})$ is again the sum of terms with alternating signs and weakly decreasing absolute value.

Step 2.

Fix a feasible and voluntary VCG mechanism defined by its rebate functions $r_p(i;a_{-i})$ in equation (9) and with efficiency loss $L(n,p)$. We prove now $L(n,p) \geq L^*(n,p)$. We also show that among the mechanisms achieving the optimal efficiency loss $L^*(n,p)$, the one given in (12) is characterized by symmetric treatment of agents and a linearity property.

The systems of inequalities characterizing F and VP are respectively

$$\sum_{i \in N} r_p(i;a_{-i}) \leq \sum_{i \in N} v_p(a_{-i}) - (n-1)v_p(a) = pa^{*(p+1)} \text{ for all } a \in \mathbb{R}_+^N \quad (26)$$

and

$$r_p(i;a_{-i}) \geq 0 \text{ for all } a \in \mathbb{R}_+^N \text{ all } i \in N \quad (27)$$

By definition of $L(n,p) = \lambda$ we have

$$\sum_{i \in N} r_p(i;a_{-i}) \geq pa^{*(p+1)} - \lambda v_p(a) = pa^{*(p+1)} - \lambda(\sum_{k=1}^{p} a^{*k}) \text{ all } a \in \mathbb{R}_+^N \quad (28)$$

In the three linear systems above, the right hand side is symmetric in all variables. Thus if n functions $r_p(i;a_{-i})$ meet all three systems, we construct a symmetric solution of these systems by setting $\widetilde{r}_p(a_{-i}) = \frac{1}{n!}\sum_{i \in N,\sigma} r_p(i;a_{-i}^{\sigma})$, where σ runs over the permutations of $N \diagdown \{i\}$ and a_{-i}^{σ} obtains by permuting the coordinates of a_{-i} accordingly. Note that $\widetilde{r}$ is symmetric in its $n-1$ variables.

Define for all $k = 0, \cdots, n-1$ the profiles $e^k = (1, \cdots, 1, 0, \cdots, 0) \in \mathbb{R}_+^n$ and $\varepsilon^k = (1, \cdots, 1, 0, \cdots, 0) \in \mathbb{R}_+^{n-1}$ such that $\sum_{j=1}^n e_i^k = \sum_{j=1}^{n-1} \varepsilon_i^k = k$, as well as $e^n = (1, \cdots, 1)$. Setting $\widetilde{r}_p(\varepsilon^k) = \rho_k$ we apply the system (26)(28) to the profile e^k, for which $(e^k)^{*(p+1)} = 0$ if $k \leq p$, $= 1$ if $k \geq p+1$, and $\sum_{k'=1}^p (e^k)^{*k'} = \min\{k,p\}$:

$$-\lambda k \leq (n-k)\rho_k + k\rho_{k-1} \leq 0 \text{ if } 0 \leq k \leq p \quad (29)$$

$$p(1-\lambda) \leq (n-k)\rho_k + k\rho_{k-1} \leq p \text{ if } p+1 \leq k \leq n \tag{30}$$

with the conventions $\rho_{-1} = \rho_n = 0$. Moreover (27) imposes $\rho_k \geq 0$ for all $k = 0, \cdots, n$.

For $k = 0$, (29) gives $\rho_0 = 0$ (budget balance at the null profile); for $k = 1$, (29) and $\rho_k \geq 0$ imply $\rho_1 = 0$, and by induction $\rho_k = 0$ for $k = 0, \cdots, p$. We are left with system (30), in which the first and last inequalities are

$$p(1-\lambda) \leq (n-p-1)\rho_{p+1} \leq p \text{ and } p(1-\lambda) \leq n\rho_{n-1} \leq p$$

We rewrite (30) in matrix form as

$$p(1-\lambda)E \leq MR \leq pE \tag{31}$$

where

$$E = \begin{bmatrix} 1 \\ \cdots \\ 1 \end{bmatrix} \in \mathbb{R}^{n-p}, R = \begin{bmatrix} \rho_{p+1} \\ \cdots \\ \rho_{n-1} \end{bmatrix} \in \mathbb{R}^{n-p-1}$$

and the $(n-p) \times (n-p-1)$ matrix M is

$$M = \begin{bmatrix} n-p-1 & 0 & 0 & .. & 0 & 0 \\ p+2 & n-p-2 & 0 & .. & 0 & 0 \\ 0 & p+3 & n-p-3 & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & .. & n-2 & 2 & 0 \\ 0 & 0 & .. & 0 & n-1 & 1 \\ 0 & 0 & .. & 0 & 0 & n \end{bmatrix}$$

The key observation is that the range $\mathcal{Z} = M(\mathbb{R}^{n-p-1})$ of M is the following hyperplane of $\mathbb{R}^{n-p}$:

$$X \in \mathcal{Z} \Leftrightarrow \binom{n}{p+1}X_1 + \binom{n}{p+3}X_3 + \cdots = \binom{n}{p+2}X_2 + \binom{n}{p+4}X_4 + \cdots$$

where if $n-p$ is odd the sum over odd coordinates of X runs until $\binom{n}{n}$, that over even coordinates runs until $\binom{n}{n-1}$, and vice versa if $n-p$ is even. The verification of this fact is straightforward. Apply now inequalities (31)

$$\begin{aligned} p(1-\lambda)\sum_{k=1,3,\cdots}\binom{n}{p+k} &\leq \sum_{k=1,3,\cdots}\binom{n}{p+k}(MR)_k \\ &= \sum_{k=2,4,\cdots}\binom{n}{p+k}(MR)_k \leq p\sum_{k=2,4,\cdots}\binom{n}{p+k} \end{aligned} \tag{32}$$

and the simple combinatorial identity

$$\sum_{k=1,3,\cdots}\binom{n}{p+k} = \binom{n-1}{p} + \sum_{k=2,4,\cdots}\binom{n}{p+k}$$

to conclude

$$L(n,p) = \lambda \geq \frac{\binom{n-1}{p}}{\sum_{k=1,3,\cdots} \binom{n}{p+k}} = \frac{\binom{n-1}{p}}{\sum_{k=0}^{n-p-1} \binom{n-1}{p+k}} = L^*(n,p).$$

We explain finally in what sense the mechanism r_p^* ((12)) is characterized by the optimality of its efficiency loss, symmetry and a linearity property. Pick an arbitrary symmetric feasible and voluntary mechanism defined by r_p in (9), such that $L(n,p) = L^*(n,p)$. The argument above shows that all inequalities in (32) are equalities, implying $(MR)_k = p(1 - L^*(n,p))$ for $k = 1, 3, \cdots$ and $(MR)_k = p$ for $k = 2, 4, \cdots$. Since M is one-to-one from $\mathbb{R}^{n-p-1}$ to $\mathcal{Z}$, this means $\rho_k = r_p(\varepsilon^k) = r_p^*(\varepsilon^k)$ for $k = p+1, \cdots, n$. As $\rho_k = 0$ for $k = 0, \cdots, p$ (by F and VP), we see that $r_p(\varepsilon^k) = r_p^*(\varepsilon^k)$ for all k.

By symmetry r_p is entirely described by its restriction to the cone $\Gamma = \{b \in \mathbb{R}^{n-1} | b_1 \geq b_2 \geq \cdots \geq b_{n-1}\}$. Any $b \in \Gamma$ is uniquely written as a non negative linear combination of the vectors $\varepsilon^k, k = 1, \cdots, n-1$, and in Γ the function r_p^* is linear with respect to non negative combinations, so it is determined by its values at the profiles $e^k, k = 1, \cdots, n-1$. This proves the claim.

9.2 Theorem 2

Step 1. We check that (15) defines a feasible mechanism with budget loss $\widehat{\Delta}$ in (16) and efficiency loss $\widehat{L}(n,p)$ in (14). As in step 1 of the previous proof we compute the sum $\sum_i \widehat{r}_p(a_{-i}) = \sum_{k=1}^n \alpha_k a^{*k}$ and we check first that $pa^{*(p+1)} - \sum_{k=1}^n \alpha_k a^{*k}$ is given by (16). For $k \geq p+2$ we have

$$\alpha_k = (k-1)(-1)^{k-p-2} \frac{p\widehat{L}(n,p)}{(k-1)L^*(n,k-1)} + (n-k)(-1)^{k-p-1} \frac{p\widehat{L}(n,p)}{kL^*(n,k)}$$

$$= (-1)^{k-p-2} p\widehat{L}(n,p)(\frac{1}{L^*(n,k-1)} - \frac{n-k}{kL^*(n,k)}) = (-1)^{k-p-2} p\widehat{L}(n,p)$$

(we omit the similar computation for $a^{*(p+1)}$). For $k \leq p$ with $p-k$ odd we get

$$\alpha_k = \widehat{L}(n,p)\{(k-1)\gamma_{k-1} + (n-k)\gamma_k\}$$

$$= \widehat{L}(n,p)\{\frac{n}{n-1}[(k-1)\frac{B_{n-2}^{(n-k+1)\rightarrow}}{\binom{n-2}{n-k}} - (n-k)\frac{B_{n-2}^{(n-k)\rightarrow}}{\binom{n-2}{n-k-1}}] - \frac{n-k}{n-1}\} = -k$$

with the help of the identity $\frac{k-1}{\binom{n-2}{n-k}} = \frac{n-k}{\binom{n-2}{n-k-1}}$. When $p-k$ is even, a similar computation gives $\alpha_k = k-1$. It is now routine to check that the budget imbalance $pa^{*(p+1)} - \sum_{k=1}^n \alpha_k a^{*k}$ icoincides with $\widehat{\Delta}(a)$ in (16).

Feasibility is clear because if $k \leq p-1$, the term $a^{*(p-k)} - a^{*(p-k+1)}$ is non negative, and the terms in the sum $\sum_{k=p+1}^n (-1)^{k-p-1} a^{*k}$ diminish in absolute value. Finally (16) implies for all $k = 1, \cdots n$:

$$\frac{\widehat{\Delta}(e^k)}{\widehat{L}(n,p)} = \min\{k,p\} \text{ if } p-k \text{ is odd, } = 0 \text{ if } p-k \text{ is even} \Rightarrow \frac{\widehat{\Delta}(e^k)}{\widehat{L}(n,p)} \leq v_p(e^k)$$

Both functions $\widehat{\Delta}(a)$ and $v_p(a)$ are symmetric in a and linear with respect to non negative combinations on the cone $\Theta = \{a \in \mathbb{R}^n_+ | a_1 \geq a_2 \geq \cdots \geq a_n\}$. Therefore $\widehat{\Delta}(a) \leq \widehat{L}(n,p) v_p(a)$ holds for all $a \in \mathbb{R}^n_+$. Moreover it is an equality for $a = e^{p+1}$ concluding the proof that the efficiency loss of our mechanism is $\widehat{L}(n,p)$.

Step 2.

Fix a feasible mechanism with rebate functions $r(i; a_{-i})$ in (9) and efficiency loss $L(n,p) = \lambda$ ((6)). We prove $\lambda \geq \widehat{L}(n,p)$.

We simply follow step 2 of the previous proof, including the symmetrization argument, until we reach the system of inequalities (29),(30) that the coefficients $\rho_k = r(\varepsilon^k), k = 0, \cdots, n-1$ must satisfy. As $r(\varepsilon^k)$ may be positive or negative, the only value of ρ_k determined by (29) is $\rho_0 = 0$. We rewrite system (29),(30) in matrix form as

$$F \leq PR \leq G \tag{33}$$

where the columns $R \in \mathbb{R}^{n-1}$, $F, G \in \mathbb{R}^n$ and the matrix P are defined as

$$R_k = \rho_k \text{ for } k = 1, \cdots, n-1$$

$$F_k = -\lambda k, G_k = 0 \text{ for } k = 1, \cdots, p$$

$$F_k = (1-\lambda)p, G_k = p \text{ for } k = p+1, \cdots, n$$

$$P = \begin{bmatrix} n-1 & 0 & 0 & 0 & 0 \\ 2 & n-2 & 0 & 0 & 0 \\ 0 & 3 & n-3 & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & .. & n-1 & 1 \\ 0 & 0 & .. & 0 & n \end{bmatrix}$$

The range $\mathcal{Z} = P(\mathbb{R}^{n-1})$ is the following hyperplane of $\mathbb{R}^n$:

$$X \in \mathcal{Z} \Leftrightarrow \binom{n}{1} X_1 + \binom{n}{3} X_3 + \cdots = \binom{n}{2} X_2 + \binom{n}{4} X_4 + \cdots$$

where each sum runs until $\binom{n}{n}$ or $\binom{n}{n-1}$ depending on the parity of n.

We distinguish two cases, according to the parity of p.

Case 1: p is even. Inequalities (33) imply (as in (32)):

$$\sum_{k=1,3,\cdots} \binom{n}{k} F_k \leq \sum_{k=1,3,\cdots} \binom{n}{k} (PR)_k = \sum_{k=2,4,\cdots} \binom{n}{k} (PR)_k \leq \sum_{k=2,4,\cdots} \binom{n}{k} G_k$$

$$\Leftrightarrow p\{ \sum_{k\ odd}^{p+1 \leq k \leq n} \binom{n}{k} - \sum_{k\ even}^{p+1 \leq k \leq n} \binom{n}{k} \} \leq \lambda \{ \sum_{k\ odd}^{1 \leq k \leq p} k \binom{n}{k} + p \sum_{k\ odd}^{p+1 \leq k \leq n} \binom{n}{k} \}$$

From $k\binom{n}{k} = n\binom{n-1}{k-1}$ and p even we get $\sum_{k\ odd}^{1 \leq k \leq p} k\binom{n}{k} = nB_{n-2}^{\rightarrow(p-2)}$. From p even again

$$\sum_{k\ odd}^{p+1 \leq k \leq n} \binom{n}{k} = B_{n-1}^{p\rightarrow}; \sum_{k\ even}^{p+1 \leq k \leq n} \binom{n}{k} = B_{n-1}^{(p+1)\rightarrow}$$

and the inequality $\lambda \geq \widehat{L}(n,p)$ follows.
Case 2: p is odd. We deduce from (33)

$$\sum_{k=2,4,\cdots} \binom{n}{k} F_k \leq \sum_{k=2,4,\cdots} \binom{n}{k} (PR)_k = \sum_{k=1,3,\cdots} \binom{n}{k} (PR)_k \leq \sum_{k=1,3,\cdots} \binom{n}{k} G_k$$

and from there the computations are entirely similar to those for the case p even, taking into account

$$\sum_{k \text{ } odd}^{p+1 \leq k \leq n} \binom{n}{k} = B_{n-1}^{(p+1)\rightarrow}; \sum_{k \text{ } even}^{p+1 \leq k \leq n} \binom{n}{k} = B_{n-1}^{p\rightarrow};$$

$$\sum_{k \text{ } even}^{1 \leq k \leq p} k \binom{n}{k} = n B_{n-2}^{\rightarrow (p-2)} \text{ if } p \geq 3, = 0 \text{ if } p = 1$$

The argument showing that the mechanism $\widehat{r}_p$ defined by (15) is uniquely optimal given symmetry and linearity in the a^{*k} is exactly as in Theorem 1. The system

$$(PR)_k = F_k \text{ for } k \text{ odd; } (PR)_k = G_k \text{ for } k \text{ even} \tag{34}$$

has a unique solution ρ_k, $k = 1, \cdots, n-1$. Together with $\rho_0 = 0$, this determines $r_p(\varepsilon^k)$ for $k = 0, \cdots, n-1$. Then we extend r_p by linearity to the cone Γ and by symmetry to the entire positive orthant $\mathbb{R}^n_+$.

9.3 Theorem 3

9.3.1 Preliminary result

If s, t are two positive integers such that $2t \geq s$, we have

$$B_s^{t\rightarrow} \leq \frac{t+1}{2t-s+1} \binom{s}{t} \tag{35}$$

Compute for all $k = 1, \cdots, t$:

$$\binom{s}{t+k} = \frac{s-t-k+1}{t+k} \binom{s}{t+k-1} \leq \frac{s-t}{t+1} \binom{s}{t+k-1}$$

$$\Rightarrow \binom{s}{t+k} \leq (\frac{s-t}{t+1})^k \binom{s}{t}$$

Setting $z = \frac{s-t}{t+1}$, which by assumption is strictly smaller than 1, we deduce

$$B_s^{t\rightarrow} \leq (\sum_{k=0}^{s-t} z^k) \binom{s}{t} \leq \frac{1}{1-z} \binom{s}{t}$$

as claimed.

9.3.2 Statement i)

Step 1: $L^*(n,p)$ decreases strictly in n:

$$\frac{B_{n-1}^{p\to}}{\binom{n-1}{p}} < \frac{B_n^{p\to}}{\binom{n}{p}} \Leftrightarrow \frac{n}{n-p}B_{n-1}^{p\to} < B_n^{p\to} = 2B_{n-1}^{p\to}+\binom{n-1}{p-1} \Leftrightarrow \frac{2p-n}{n-p}B_{n-1}^{p\to} < \binom{n-1}{p-1}$$

This is clear if $2p \leq n$ as the left hand side is non positive. If $2p > n$, (35) gives the upper bound $B_{n-1}^{p\to} \leq \frac{p+1}{2p-n+2}\binom{n-1}{p}$ so the desired inequality follows from

$$\frac{2p-n}{n-p}\frac{p+1}{2p-n+2}\binom{n-1}{p} < \binom{n-1}{p-1} \Leftrightarrow \frac{2p-n}{2p-n+2} < \frac{p}{p+1}$$

which is easily checked.

Step 2: $L^*(n,p) = \frac{\binom{n-1}{p}}{B_{n-1}^{p\to}}$ increases strictly in p:

$$\frac{B_{n-1}^{(p-1)\to}}{\binom{n-1}{p-1}} > \frac{B_{n-1}^{p\to}}{\binom{n-1}{p}} \Leftrightarrow \frac{p}{n-p}B_{n-1}^{p\to} < B_{n-1}^{(p-1)\to} \Leftrightarrow \frac{2p-n-2}{n-p}B_{n-1}^{p\to} < \binom{n-1}{p-1} \tag{36}$$

The right hand side inequality is true if $2p \leq n$, so we assume $2p > n$ and we majorize $B_{n-1}^{p\to}$ with the help of (35); now the desired inequality follows from

$$\frac{2p-n-2}{2p-n}\frac{p+1}{n-p}\binom{n-1}{p} < \binom{n-1}{p-1} \Leftrightarrow \frac{2p-n-2}{2p-n} < \frac{p}{p+1}$$

and the proof is complete.

Step 3 $\widehat{L}(n,p)$ is single-peaked in n. The inequality $\widehat{L}(n,p) < \widehat{L}(n+1,p)$ amounts to

$$n(B_{n-1}^{p\to} + \frac{n}{p}B_{n-2}^{\to(p-2)}) > (n-p)(B_n^{p\to} + \frac{n+1}{p}B_{n-1}^{\to(p-2)})$$

Taking into account $B_n^{p\to} = 2B_{n-1}^{p\to} + \binom{n-1}{p-1}$ and $B_{n-1}^{\to(p-2)} = 2B_{n-2}^{\to(p-2)} - \binom{n-2}{p-2}$, this is rearranged as

$$(2p-n)B_{n-1}^{p\to} + ((2p-n)\frac{n+1}{p} - \frac{n}{p})B_{n-2}^{\to(p-2)} + \frac{2p-n-1}{p}\binom{n-2}{p-1} > 0$$

Clearly this inequality fails if $n \geq 2p$, and holds if $n \leq 2p-1$, as was to be proved.

Step 4 $\widehat{L}(n,p)$ is single-peaked in p. The inequality $\widehat{L}(n,p) < \widehat{L}(n,p+1)$ amounts to

$$(n-p-1)(B_{n-1}^{p\to} + \frac{n}{p}B_{n-2}^{\to(p-2)}) > (p+1)(B_{n-1}^{(p+1)\to} + \frac{n}{p+1}B_{n-2}^{\to(p-1)})$$

which simplifies to

$$(n-2p-2)B_{n-1}^{p\to} + (n-2p-1)\frac{n}{p}B_{n-2}^{\to(p-2)} + \binom{n-2}{p} > 0$$

This inequality holds if $2p \leq n-2$ and fails if $2p \geq n-1$, so the peak of $\widehat{L}(n,p)$ is at $\{\frac{n}{2}\}$ as claimed.

9.3.3 Statement ii)

By statement $i)$ we only need to prove $\widehat{L}(n,\{\frac{n}{2}\}) \leq \frac{4}{3\sqrt{n}}$ for all $n \geq 3$. The proof uses the inequality $\binom{n}{\{\frac{n}{2}\}} \leq \frac{2^n}{\sqrt{n}}$ for all $n \geq 2$. It follows from the slightly stronger inequality $\binom{n}{\{\frac{n}{2}\}} \leq \frac{n}{n+1}\frac{2^n}{\sqrt{n}}$ for all $n \geq 3$, which is easily proven by induction first on even numbers, then on odd ones.

Set $m = \{\frac{n}{2}\}$ for simplicity and notice that (14) implies

$$\widehat{L}(n,m) \leq \frac{\binom{n-1}{m}}{B_{n-1}^{m\rightarrow} + 2B_{n-2}^{\rightarrow(m-2)}} \text{ for all } n$$

Compute

$$B_{n-1}^{m\rightarrow} + 2B_{n-2}^{\rightarrow(m-2)} = B_{n-1}^{m\rightarrow} + B_{n-1}^{\rightarrow(m-2)} + \binom{n-2}{m-2} = 2^{n-1} - \binom{n-2}{m-1}$$

$$\Rightarrow \widehat{L}(n,m) \leq \frac{\binom{n-1}{m}}{2^{n-1} - \binom{n-2}{m-1}}$$

If n is even, $m = \frac{n}{2}$ and this inequality is in fact an equality. We have:

$$\widehat{L}(n,\frac{n}{2}) = \frac{\frac{1}{2}\binom{n}{m}}{2^{n-1} - \binom{n-2}{m-1}} \leq \frac{\frac{2^{n-1}}{\sqrt{n}}}{2^{n-1} - \frac{2^{n-2}}{\sqrt{n-2}}} = \frac{1}{\sqrt{n}}\frac{1}{1 - \frac{1}{2\sqrt{n-2}}} \leq \frac{4}{3\sqrt{n}}$$

where the last inequality holds if $n \geq 6$. For $n = 4$ we compute directly $\widehat{L}(4,2) = \frac{1}{2} \leq \frac{4}{3\sqrt{4}}$.

If n is odd, $m = \frac{n-1}{2}$ and we have

$$\widehat{L}(n,\frac{n-1}{2}) \leq \frac{\binom{n-1}{m}}{2^{n-1} - \frac{1}{2}\binom{n-1}{m}} \leq \frac{\frac{2^{n-1}}{\sqrt{n-1}}}{2^{n-1} - \frac{2^{n-2}}{\sqrt{n-1}}} = \frac{1}{\sqrt{n-1}}\frac{1}{1 - \frac{1}{2\sqrt{n-1}}} \leq \frac{4}{3\sqrt{n}}$$

where the last inequality holds if $n \geq 5$. For $n = 3$ we compute directly $\widehat{L}(3,1) = \frac{2}{3} \leq \frac{4}{3\sqrt{3}}$.

9.3.4 Statements iii)

Fix p then as n increases, $\binom{n-1}{p} \simeq \frac{n^p}{p!}$. Moreover $B_{n-1}^{p\rightarrow} \simeq 2^{n-1}$ because $B_{n-1}^{p\rightarrow} = 2^{n-1} - B_{n-1}^{\rightarrow(p-1)}$ and $B_{n-1}^{\rightarrow(p-1)}$ is a polynomial of degree $p-1$. Finally $\frac{n}{p}B_{n-2}^{\rightarrow(p-2)}$ is a polynomial of degree $p-2$. The statement follows at once.

9.3.5 Statement iv)

Fix K and a sequence p_n as in the premises of (17). Stirling's formula $n! \simeq (\frac{n}{e})^n\sqrt{2\pi n}$ implies easily that

$$\binom{n-1}{\frac{n}{2}-K} \simeq \binom{n-1}{\frac{n}{2}+K} \simeq \binom{n-1}{p_n} \simeq \frac{2^n}{\sqrt{2\pi n}}$$

Then $(B_{n-1}^{(\frac{n}{2}-K)\to} - B_{n-1}^{(\frac{n}{2}+K)\to}) \simeq C\frac{2^n}{\sqrt{n}}$ for some constant C, hence

$$B_{n-1}^{(\frac{n}{2}+K)\to} \simeq B_{n-1}^{(\frac{n}{2}-K)\to} \simeq B_{n-1}^{p_n\to} \simeq 2^{n-2} \text{ and } \frac{n}{p_n}B_{n-2}^{\to(p_n-2)} \simeq 2^{n-2}$$

The desired estimates follow.

9.3.6 Statements v)

Property (18). We fix δ and a sequence p_n as in the premises of (18) and we compute

$$L^*(n,p_n) = \frac{\binom{n-1}{p_n}}{2^{n-1} - B_{n-1}^{(n-p_n)\to}} = \frac{1}{\frac{2^{n-1}}{\binom{n-1}{p_n}} - \frac{B_{n-1}^{(n-p_n)\to}}{\binom{n-1}{p_n}}}$$

Inequality (35) applies to $B_{n-1}^{(n-p_n)\to}$ because $p_n < \frac{n}{2}$ and it gives

$$\frac{B_{n-1}^{(n-p_n)\to}}{\binom{n-1}{p_n}} = \frac{p_n}{n-p_n}\frac{B_{n-1}^{(n-p_n)\to}}{\binom{n-1}{n-p_n}} \leq \frac{B_{n-1}^{(n-p_n)\to}}{\binom{n-1}{n-p_n}} \leq \frac{n-p_n+1}{n-2p_n+2} \leq \frac{n-\delta n+1}{n-2\delta n+2} \leq \frac{1-\delta}{1-2\delta}$$

Thus it is enough to check that $\frac{2^{n-1}}{\binom{n-1}{p_n}}$ grows like $\frac{1}{\alpha^n}$ for some $\alpha < 1$. This is again a consequence of the Stirling formula. Write $\lfloor \delta n \rfloor$ for the largest integer below δn and compute

$$\binom{n-1}{p_n} \leq \binom{n-1}{\lfloor \delta n \rfloor} = \frac{n - \lfloor \delta n \rfloor}{n}\binom{n}{\lfloor \delta n \rfloor} \simeq (1-\delta)\frac{n!}{\lfloor \delta n \rfloor!(n-\lfloor \delta n \rfloor)!}$$

$$\simeq (1-\delta)\frac{\sqrt{2\pi n}(\frac{n}{e})^n}{\sqrt{2\pi\delta n}(\frac{\delta n}{e})^{\delta n}\sqrt{2\pi(1-\delta)n}(\frac{(1-\delta)n}{e})^{(1-\delta)n}} = \sqrt{\frac{(1-\delta)}{2\pi\delta}}\frac{1}{\sqrt{n}(\delta^\delta(1-\delta)^{(1-\delta)})^n}$$

Now we have

$$\frac{2^{n-1}}{\binom{n-1}{p_n}} \geq \frac{2^{n-1}}{\binom{n-1}{\lfloor \delta n \rfloor}} \simeq \sqrt{\frac{\pi\delta}{2(1-\delta)}}\sqrt{n}(2\delta^\delta(1-\delta)^{(1-\delta)})^n$$

and the desired conclusion follows from the inequality $2\delta^\delta(1-\delta)^{(1-\delta)} > 1$ whenever $\delta \neq \frac{1}{2}$. As $\widehat{L}(n,p) \leq L^*(n,p)$ the proof of (18) is complete.

Note that we proved a slightly stronger statement than (18), namely an upper bound of $L^*(n,p_n)$ is equivalent to $\sqrt{\frac{2(1-\delta)}{\pi\delta}}\frac{1}{\sqrt{n}(2\delta^\delta(1-\delta)^{(1-\delta)})^n}$.

Property (19)

We fix a sequence p_n as in the premises of (19) and prove first the lower bound on $L^*(n, p_n)$. Write $\lceil \delta n \rceil$ for the smallest integer above δn. As $2\lceil \delta n \rceil \geq n - 1$ we can apply inequality (35) to $B_{n-1}^{\lceil \delta n \rceil \rightarrow}$:

$$B_{n-1}^{\lceil \delta n \rceil \rightarrow} \leq \frac{\lceil \delta n \rceil + 1}{2\lceil \delta n \rceil - n + 2} \binom{n-1}{\lceil \delta n \rceil}$$

Now use the monotonicity of $L^*(n, p)$ in p and compute

$$L^*(n, p_n) \geq L^*(n, \lceil \delta n \rceil) = \frac{\binom{n-1}{\lceil \delta n \rceil}}{B_{n-1}^{\lceil \delta n \rceil \rightarrow}} \geq \frac{2\lceil \delta n \rceil - n + 2}{\lceil \delta n \rceil + 1} \geq \frac{2\delta n - n + 2}{\delta n + 2} \geq \frac{2\delta - 1}{\delta}$$

as desired. Next we check that $\widehat{L}(n, p_n)$ is exponential in n:

$$\widehat{L}(n, p_n) \leq \frac{\binom{n-1}{p_n}}{\frac{n}{p_n} B_{n-2}^{\rightarrow(p_n - 2)}} \leq \frac{\binom{n-1}{\lceil \delta n \rceil}}{B_{n-2}^{\rightarrow(\lceil \delta n \rceil - 2)}} = \frac{\binom{n-1}{\lceil \delta n \rceil}}{2^{n-2} - B_{n-2}^{(\lceil \delta n \rceil - 1)\rightarrow}}$$

and from there the argument is exactly as in the proof of property (18): because $2(\lceil \delta n \rceil - 1) \geq n - 2$ we can use (35) to get a finite upper bound for $\frac{B_{n-2}^{(\lceil \delta n \rceil - 1)\rightarrow}}{\binom{n-1}{\lceil \delta n \rceil}}$, then we check that $\frac{2^{n-2}}{\binom{n-1}{\lceil \delta n \rceil}}$ grows exponentially in n. We omit the details.

9.4 Proposition 1

For the feasible mechanism (15), write $\widehat{h}_i(a_{-i})$ for its representation in the form (3). As its efficiency loss is $\lambda = \widehat{L}(n, p)$, we have

$$(n-1)v_p(a) \leq \sum_{i \in N} \widehat{h}_i(a_{-i}) \leq \{(n-1) + \lambda\} v_p(a)$$

$$\Rightarrow \{(n-1) - \frac{\lambda}{2}\} v_p(a) \leq (1 - \frac{\lambda}{2(n-1)}) \sum_{i \in N} \widehat{h}_i(a_{-i}) \leq \{(n-1) + \frac{\lambda}{2}\} v_p(a)$$

therefore $h_i = (1 - \frac{\lambda}{2(n-1)})\widehat{h}_i$ defines an (unfeasible) mechanism with efficiency loss at most $\frac{\lambda}{2}$. Thus $L^{\#}(n, p) \leq \frac{1}{2}\widehat{L}(n, p)$. Conversely pick a mechanism (3) with worst relative cost $\mu = L^{\#}(n, p)$ and compute

$$\{(n-1) - \mu\} v_p(a) \leq \sum_{i \in N} h_i(a_{-i}) \leq \{(n-1) + \mu\} v_p(a)$$

$$\Rightarrow (n-1)v_p(a) \leq \frac{n-1}{n-1-\mu} \sum_{i \in N} h_i(a_{-i}) \leq \{(n-1) + \frac{2\mu}{1 - \frac{\mu}{n-1}}\} v_p(a)$$

$$\Rightarrow \lambda \leq \frac{2\mu}{1 - \frac{\mu}{n-1}} \Rightarrow \mu \geq \frac{\lambda}{2 + \frac{\lambda}{n-1}} \geq \frac{1}{2 + \frac{1}{n-1}} \lambda$$

9.5 Proposition 2

The proof follows closely that of Theorem 1. Prove first equation (24) (implying feasibility) by summing up the rebates (23) and checking that $pa^{*(p+1)} - \sum_{i\in N}\widetilde{r}_p^m(a_{-i})$ is precisely the right hand side of (24). We omit this computation. To check VP$\Leftrightarrow \widetilde{r}_p^m(a_{-i}) \geq 0$, we notice that $L^*(n,k,m+p-k)$ increases in k. Indeed for all s,t', $\frac{B_s^{t,t'}}{\binom{s-1}{t-1}}$ is decreasing in t:

$$\frac{B_s^{t+1,t'}}{B_s^{t,t'}} \leq \frac{B_s^{(t+1)\rightarrow}}{B_s^{t\rightarrow}} \leq \frac{\binom{s-1}{t}}{\binom{s-1}{t-1}}$$

where the second inequality is the fact that $\frac{B_s^{t\rightarrow}}{\binom{s-1}{t-1}}$ decreases in t, as noted at the end of step 1 in subsection 9.1.

It remains to prove the expression (22) for the optimal $L^*(n,p,m)$. Fix a feasible and voluntary mechanism with rebates $r_p(i;a_{-i})$ that only records the $p+m$ highest valuations: the function $r_p(i;\cdot)$ can only depend upon the variables $a_{-i}^{*k}, k=1,\cdots,p+m-1$. We must show that the efficiency loss $L(n,p)=\lambda$ of this mechanism is not smaller than $\frac{\binom{n-1}{p}}{B_{n-1}^{p,p+m-1}}$. The symmetrization argument in step 2 of subsection 9.1 remains valid, so we need only to consider a symmetric function $r_p(a_{-i}^{*k})_{k=1,\cdots,p+m-1}$.

Setting $r_p(\varepsilon^k)=\rho_k$, we now have $\rho_{p+m-1}=\rho_{p+m}=\cdots=\rho_{n-1}$. Thus the system of linear constraints (29) on ρ_k implied by feasibility and the definition of λ (6) at the profiles e^k is unchanged, while system (30) becomes

$$\begin{gathered}p(1-\lambda) \leq (n-k)\rho_k + k\rho_{k-1} \leq p \text{ if } p+1 \leq k \leq p+m-1\\ \text{and } p(1-\lambda) \leq n\rho_{p+m-1} \leq p\end{gathered}$$

As in Theorem 1, VP imposes $\rho_k \geq 0$ for all k, and in combination with (29) this gives $\rho_k = 0$ for $k=0,\cdots,p$. Thus we can write the above system for the variables $\rho_k, k=p+1,\cdots,m+p-1$ in the matrix form

$$p(1-\lambda)E' \leq M'R' \leq pE' \tag{37}$$

where

$$E' = \begin{bmatrix}1\\ \dots\\ 1\end{bmatrix} \in \mathbb{R}^m, R' = \begin{bmatrix}\rho_{p+1}\\ \dots\\ \rho_{p+m-1}\end{bmatrix} \in \mathbb{R}^{m-1}$$

and with the notation $q=p+m-1$, the $m\times(m-1)$ matrix M' is

$$M' = \begin{bmatrix}n-p-1 & 0 & 0 & .. & 0 & 0\\ p+2 & n-p-2 & 0 & .. & 0 & 0\\ 0 & p+3 & n-p-3 & .. & 0 & 0\\ .. & .. & .. & .. & .. & ..\\ 0 & 0 & .. & q-1 & n-q+1 & 0\\ 0 & 0 & .. & 0 & q & n-q\\ 0 & 0 & .. & 0 & 0 & n\end{bmatrix}$$

The range $\mathcal{Z}'=M'(\mathbb{R}^{m-1})$ of M' is the following hyperplane of $\mathbb{R}^m$:

$$X \in \mathcal{Z}' \Leftrightarrow \binom{n}{p+1}X_1 + \binom{n}{p+3}X_3 + \cdots = \binom{n}{p+2}X_2 + \binom{n}{p+4}X_4 + \cdots$$

where the last two terms of the sum containing the X_m term (the sum of odd coordinates of X if m is odd, of even coordinates if m is even) are $\binom{n}{q-1}X_{m-2} + \binom{n-1}{q}X_m$, and the last term of the other sum is $\binom{n}{q}X_{m-1}$. Note that the coefficient of X_m is $\binom{n-1}{q}$ while all other terms are of the form $\binom{n}{\cdot}$. We omit the straightforward proof of this claim. Apply now inequalities (37)

$$\begin{aligned} p(1-\lambda)\sum_{k=1,3,\cdots}\binom{n}{p+k} &\leq \sum_{k=1,3,\cdots}\binom{n}{p+k}(M'R')_k \\ &= \sum_{k=2,4,\cdots}\binom{n}{p+k}(M'R')_k \leq p\sum_{k=2,4,\cdots}\binom{n}{p+k} \end{aligned} \tag{38}$$

where the sums $S = \sum_{k=1,3,\cdots}\binom{n}{p+k}$ and $T = \sum_{k=2,4,\cdots}\binom{n}{p+k}$ run until m or $m-1$ and the coefficient of X_m is as explained above. Check now that irrespective of the parity of m

$$\sum_{k=1,3,\cdots}\binom{n}{p+k} = \sum_{k=0}^{q}\binom{n-1}{p+k}; \sum_{k=2,4,\cdots}\binom{n}{p+k} = \sum_{k=1}^{q}\binom{n-1}{p+k}$$

to conclude from (38)

$$L(n,p,m) = \lambda \geq \frac{\binom{n-1}{p}}{\sum_{k=0}^{q}\binom{n-1}{p+k}} = \frac{\binom{n-1}{p}}{B_{n-1}^{p,q}} = L^*(n,p,m)$$

9.5.1 $L^*(n,p,m)$ decreases strictly in n

Set $q = p+m-1 \geq p$. By equation (22) we must show that $\frac{B_{n-1}^{p,q}}{\binom{n-1}{p}}$ increases strictly in n. This amounts to

$$nB_{n-1}^{p,q} < (n-p)B_n^{p,q} \Leftrightarrow nB_{n-1}^{p,q} < (n-p)(2B_{n-1}^{p,q} + \binom{n-1}{p-1} - \binom{n-1}{q})$$

$$\Leftrightarrow (2p-n)B_{n-1}^{p,q} + (n-p)\binom{n-1}{q} < (n-p)\binom{n-1}{p-1}$$

We let the reader check that the left-hand side of the above inequality decreases strictly in q, and that for $q = p$ it is an equality.